

# Application of the Real Space Dynamic Renormalization Group Method to the One-Dimensional Spin-Exchange Model

Gene F. Mazenko<sup>1</sup> and Enis Oğuz<sup>1</sup>

*Received July 21, 1981*

---

We apply the real space dynamic renormalization group method to the one-dimensional spin-exchange kinetic Ising model. We show that the conservation of magnetization property of this model is preserved directly under renormalization. We also demonstrate that one can derive recursion relations for the space- and time-dependent correlation functions and that the iterated solutions of these recursion relations lead to the appropriate hydrodynamic forms in the small-wavenumber and -frequency regime.

---

**KEY WORDS:** Real space dynamic renormalization group; conserved magnetization; kinetic Ising models.

## 1. INTRODUCTION

The real space dynamic renormalization group (RSDRG) method has recently been introduced and applied<sup>(1-16),2</sup> to one- and two-dimensional spin-flip kinetic Ising models. In particular in our own work it was shown<sup>(2,5,8,16)</sup> how one could map the pseudo-Liouville operators governing the dynamics in these models onto coarse-grained operators governing the dynamics of a set of coarse-grained spins in such a way that the "renormalized" operator is free from any pathological long-range interactions in space or time. We also showed<sup>(12-16)</sup> that one can derive recursion relations relating space- and time-dependent correlation functions defined

---

<sup>1</sup> The James Franck Institute and the Department of Physics, The University of Chicago, Chicago, Illinois 60637.

<sup>2</sup> References 1-11 focus primarily on the determination of the dynamical critical index  $z$ . In Refs. 12-16 the emphasis is on the determination of time-space-dependent correlation functions over a wide range of parameters.

for an original set of spins to a coarse-grained set of block spins. The physics behind this development is that short range rapidly varying degrees of freedom can be treated directly using perturbation theory while the long-range, slowly varying degrees of freedom are treated iteratively. In this way we simultaneously preserve the local structure and build up long-range correlations. The iterated solutions of these recursion relations have led to excellent approximations for various static and dynamic quantities in the spin-flip models over a wide range of temperatures, wave numbers, and frequencies.

Spin-exchange kinetic Ising models<sup>(17-19)</sup> are constructed such that the total magnetization is a conserved quantity. While these models have been more extensively studied numerically<sup>(20,21)</sup> than the single-spin-flip models, there is in greater than one dimension almost no analytical work beyond Kawasaki's<sup>(17,18)</sup> original mean-field-like treatment of diffusion. Recently there has been some discussion of general spin-exchange models in one dimension<sup>(23)</sup> and Zwerger<sup>(19)</sup> has found a particular model for which one can find the diffusion coefficient exactly. We discuss this further in Section 2. However, there has been very little work treating models which cannot be solved exactly. Developing approximation methods for such problems in one dimension seems worthwhile since it is very unlikely that we will find an exact solution in higher dimensions. With this in mind we address three basic questions in this paper:

i. Can our RSDRG methods be extended to handle conserved variables?

ii. Will our recursion relations connecting dynamic structure factors on the coarse-grained and original lattices be able to reproduce the appropriate hydrodynamical spectrum in the small-frequency and small-wave-number limits?

iii. Will our coarse-graining procedure, where we must construct the mapping function  $T[\mu | \sigma]$  as a solution to an eigenvalue equation, lead to a different treatment of the static behavior than we found in the single-spin-flip case? The point of this question is whether or not our method couples the treatment of the static behavior to that of the dynamic behavior.

The answers to (i) and (ii) are an unqualified yes. In addressing question (iii) we found exactly the same static behavior as for the single-spin-flip case to the order we investigated. We do not have a general proof that the static behavior can be treated in an identical manner for the two models to all orders in our RSDRG development.

In Section 2 we introduce the model studied and point out a few quantities that are known exactly. We describe briefly the RSDRG method in Section 3 and indicate how it can be directly applied to a system with a conserved variable. In Section 4 we develop a perturbation theory approach

for implementing the RSDRG where we break the system up into cells of spins and treat the interactions between cells as a small parameter. We implement this expansion in Section 5 to carry out our renormalization procedure to lowest order in the expansion parameter. In Section 6 we determine the recursion relations satisfied by the basic parameters in the theory and then between various correlation functions. Using these results we tie down the parameters which enter into the perturbation expansion. Finally, in Section 7 we discuss the results of our calculations for various dynamic quantities, some of the limitations of this treatment and their possible remedies.

## 2. THE ONE-DIMENSIONAL SPIN-EXCHANGE MODEL

We consider a one-dimensional chain of  $N$  ferromagnetic Ising spins  $\{\sigma\} = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$  with lattice spacing  $a_0$ . The equilibrium probability distribution function is given by the Boltzmann factor

$$P[\sigma] = \frac{e^{H[\sigma]}}{Z} \tag{2.1}$$

where  $H[\sigma]$  is the Hamiltonian (multiplied by  $-1/k_B T$ ) given by

$$H[\sigma] = K \sum_n \sigma_n \sigma_{n+1} \tag{2.2}$$

where we consider nearest-neighbor coupling,  $K(> 0)$  is just the exchange constant divided by  $k_B T$  and the partition function  $Z$  is the sum over all spin configurations. We can, of course, calculate any static correlation function of interest exactly in this case.

The dynamics in our model is generated by a pseudo-Liouville operator  $\tilde{D}_\sigma$  such that, for example, the equilibrium correlation of a spin at lattice site  $j$  at time zero with a spin at site  $i$  at time  $t$  is given by

$$C_{ij}(t) = \sum_\sigma P[\sigma] \sigma_j e^{\tilde{D}_\sigma t} \sigma_i \tag{2.3}$$

$\tilde{D}_\sigma$  is a matrix operator ( $\tilde{D}[\sigma|\sigma]$ ) whose general properties are described in Ref. 8. There are three basic conditions that  $\tilde{D}_\sigma$  should satisfy: (i) stationarity and detailed balance,<sup>(8)</sup> (ii) locality in space (as required of a model with nearest-neighbor interactions only), and (iii) conservation of total spin.

The general one-dimensional operator which satisfies (ii) and (iii) is of the form

$$\tilde{D}_\sigma = \tilde{D}_{\alpha,\sigma} + \tilde{D}_{\beta,\sigma} \tag{2.4}$$

where

$$\tilde{D}_{\alpha,\sigma} = -\frac{\alpha}{2} \sum_i \Lambda_{\sigma,\sigma'}^{[i,i+1]} W_\alpha^{[i,i+1]}(\sigma)(\sigma_i - \sigma_{i+1})(\sigma'_i - \sigma'_{i+1}) \tag{2.5}$$

and

$$\tilde{D}_{\beta,\sigma} = -\frac{\beta}{2} \sum_i \Lambda_{\sigma,\sigma'}^{[i,i+1]} W_{\beta}^{[i,i+1]}(\sigma) \sigma_i \sigma_{i+1} \sigma'_i \sigma'_{i+1} \quad (2.6)$$

Here,  $\alpha$  and  $\beta$  are constants which set the time scales,  $\Lambda_{\sigma,\sigma'}^{[i,i+1]}$  is the product of  $\delta$  functions, which sets  $\sigma$  equal to  $\sigma'$  except at sites  $i$  and  $i+1$  and the  $W^{[i,i+1]}(\sigma)$  give the transition probabilities for the exchange processes. The general form of  $W_{\alpha}^{[i,i+1]}(\sigma)$  and  $W_{\beta}^{[i,i+1]}(\sigma)$  for an operator  $\tilde{D}_{\sigma}$  which satisfy (i) and (ii), are the same and given by

$$W^{[i,i+1]}(\sigma) = (1 + C\sigma_{i-1}\sigma_{i+2})(1 - u\sigma_{i-1}\sigma_i)(1 - u\sigma_i\sigma_{i+1})(1 - u\sigma_{i+1}\sigma_{i+2}) \quad (2.7)$$

where  $u = \tanh K$  and  $C$  is an arbitrary constant. The operator  $\tilde{D}_{\alpha,\sigma}$  is the usual spin-exchange operator (SEO).<sup>(17,19,23)</sup> The general form of  $W^{[i,i+1]}(\sigma)$  is due to Haake and Thol.<sup>(23)</sup> [Their parameter  $\beta \equiv \beta_{\text{HT}}$  is related to our parameter  $C$  through the relation  $\beta_{\text{HT}} = (C - u^2)/(1 - Cu^2)$ .]

The reason that we consider an additional operator  $\tilde{D}_{\beta,\sigma}$  is twofold:

i. It is generated by our RSDRG transformation.

ii. It eliminates an infinite set of conservation laws<sup>(23)</sup> which seem inappropriate for a stochastic dynamics. These conserved modes are given by  $\prod_{j=1}^N \sigma_j$ ,  $\sum_{i=1}^N \sigma_i \prod_{j=1}^N \sigma_j$ ,  $\sum_{i,k=i}^N \sigma_i \sigma_k \prod_{j=1}^N \sigma_j$ , etc., and they complicate any analysis in terms of a finite number of spins. One could use different choices of  $C$  for  $W_{\alpha}^{[i,i+1]}(\sigma)$  and  $W_{\beta}^{[i,i+1]}(\sigma)$ . We will, however, choose  $C = 0$  and use the same  $W^{[i,i+1]}(\sigma)$  for both operators. Thus, our operator can be written as

$$\begin{aligned} \tilde{D}_{\sigma} = & -(1/2) \sum_i \Lambda_{\sigma,\sigma'}^{[i,i+1]} W^{[i,i+1]}(\sigma) \\ & \times [\alpha(\sigma_i - \sigma_{i+1})(\sigma'_i - \sigma'_{i+1}) + \beta\sigma_i\sigma_{i+1}\sigma'_i\sigma'_{i+1}] \end{aligned} \quad (2.8)$$

where

$$W^{[i,i+1]}(\sigma) = (1 - u\sigma_{i-1}\sigma_i)(1 - u\sigma_i\sigma_{i+1})(1 - u\sigma_{i+1}\sigma_{i+2}) \quad (2.9)$$

Equations (2.3), (2.8), and (2.9) define the dynamical problem of interest. We note that the case of  $\beta = 0$ ,  $C = u^2$  in (2.6) and (2.7) has been examined by Zwerger.<sup>(19)</sup> We will compare our results with his in Section 7.

We, of course, know everything about the equilibrium ( $t = 0$ ) properties of this system. We can ask what do we know about the dynamic properties of this model? It is convenient in discussing the dynamics to introduce the Fourier-Laplace transform<sup>3</sup>

$$C(q, z) = -i \int_0^{\infty} dt e^{izt} \frac{1}{N} \sum_{nm} e^{i(n-m)q} C_{nm}(t) \quad (2.10)$$

$$= \langle \sigma_{-q} R(z) \sigma_q \rangle \quad (2.11)$$

<sup>3</sup> From here on we will omit the factor  $a_0$  in  $qa_0$ .

where

$$\sigma_q = \frac{1}{\sqrt{N}} \sum_n e^{-inq} \sigma_n \tag{2.12}$$

and  $R(z)$  is the resolvent operator

$$R(z) \equiv |z - i\tilde{D}_\sigma|^{-1} \tag{2.13}$$

We can rewrite  $C(q, z)$  in the form

$$C(q, z) = \frac{\tilde{C}(q)}{z + i\Phi(q, z)} \tag{2.14}$$

where  $\tilde{C}(q)$  is the static structure factor,<sup>(8)</sup> and  $\Phi(q, z)$  is the “memory function” associated with  $C(q, z)$ . Using standard operator identities one can decompose  $\Phi(q, z)$  into a “static” and a “dynamic” part:

$$\Phi(q, z) = \Phi^{(s)}(q) + \Phi^{(d)}(q, z) \tag{2.15}$$

where

$$\Phi^{(s)}(q)\tilde{C}(q) = -\langle \sigma_{-q} \tilde{D}_\sigma \sigma_q \rangle \tag{2.16}$$

and

$$\begin{aligned} \Phi^{(d)}(q, z)\tilde{C}(q) = & -i \left[ \langle (\tilde{D}_\sigma \sigma_{-q}) R(z) (\tilde{D}_\sigma \sigma_q) \rangle \right. \\ & \left. - \langle (\tilde{D}_\sigma \sigma_{-q}) R(z) \sigma_q \rangle \cdot C^{-1}(q, z) \cdot \langle \sigma_{-q} R(z) (\tilde{D}_\sigma \sigma_q) \rangle \right] \end{aligned} \tag{2.17}$$

Note that  $\Phi^{(d)}(q, z)$  goes to zero as  $1/z$  for large  $z$ , so  $\Phi^{(s)}(q)$  has the physical interpretation of the initial decay rate for  $C(q, t)$ . We can determine  $\Phi^{(s)}(q)$  explicitly. On applying  $\tilde{D}_\sigma$  to a single spin  $\sigma_i$  we obtain

$$\begin{aligned} \tilde{D}_\sigma \sigma_i = & -2\alpha(1+u) \{ 2\sigma_i - (1+u)(\sigma_{i-1} + \sigma_{i+1}) + u(\sigma_{i-2} + \sigma_{i+2}) \\ & + u\sigma_i [ \sigma_{i+1}(\sigma_{i-1} - \sigma_{i+2}) + \sigma_{i-1}(\sigma_{i+1} - \sigma_{i-2}) ] \\ & + u^2\sigma_{i-1}\sigma_{i+2}(\sigma_{i+1} - \sigma_i) + u^2\sigma_{i-2}\sigma_{i+1}(\sigma_{i-1} - \sigma_i) \} \end{aligned} \tag{2.18}$$

or, in terms of Fourier transforms,

$$\begin{aligned} \tilde{D}_\sigma \sigma_q = & -2\alpha(1+u) \{ 2\sigma_q [ 1 + u \cos 2q - (1+u) \cos q ] \\ & + 2u(1 - \cos q) T_q + u^2(e^{-iq} - 1) V_q \} \end{aligned} \tag{2.19}$$

$$V_q = \frac{1}{\sqrt{N}} \sum_n e^{iqn} \sigma_{n-2} (\sigma_n - \sigma_{n-1}) \sigma_{n+1} \tag{2.20}$$

$$T_q = \frac{1}{\sqrt{N}} \sum_n e^{iqn} \sigma_{n-1} \sigma_n \sigma_{n+1} \tag{2.21}$$

We see that there are nonlinear contributions to  $\tilde{D}_\sigma \sigma_q$ . One can then show that

$$\langle \sigma_{-q} \tilde{D}_\sigma \sigma_q \rangle = -4\alpha(1-u^2)^3(1-\cos q) \quad (2.22)$$

We can therefore evaluate the static part of the memory function explicitly:

$$\phi^{(s)}(q) = 8\alpha(1-u^2)^3 \sin^2(q/2) \tilde{C}^{-1}(q) \quad (2.23)$$

While we are unable to evaluate the dynamic part of the memory function explicitly we can simplify it somewhat by using its “one-body irreducible” property. That is any linear part to  $\tilde{D}_\sigma \sigma_q$  does not contribute to  $\phi^{(d)}$  at its “end points.” We can write

$$\begin{aligned} \phi^{(d)}(q, z) \tilde{C}(q) &= -i8\alpha^2(1+u^2)u^2(1-\cos q) \\ &\times \{2\phi_{TT}(q, z)(1-\cos q) + u(e^{-iq} - 1)\phi_{TV}(q, z) \\ &\quad + u(e^{iq} - 1)\phi_{VT}(q, z) + u^2\phi_{VV}(q, z)\} \end{aligned} \quad (2.24)$$

where

$$\phi_{AB}(q, z) = \langle A_{-q} R(z) B_q \rangle - \langle A_{-q} R(z) \sigma_q \rangle C^{-1}(q, z) \langle \sigma_{-q} R(z) B_q \rangle \quad (2.25)$$

We can now check several important limits. For small wave numbers and frequencies we expect the problem to be dominated by the hydrodynamic mode associated with the conserved spin. That is, we expect the correlation function to be of the form

$$C(q, z) = \frac{\tilde{C}(0)}{z + iDq^2} \quad (2.26)$$

where  $D$  is the spin-diffusion coefficient. Comparing (2.26) with (2.14) we can read-off the formal expression

$$D = \lim_{z \rightarrow 0} \lim_{q \rightarrow 0} \frac{1}{q^2} \phi(q, z) \quad (2.27)$$

We have written  $\phi^{(s)}$  and  $\phi^{(d)}$  in such a way that we can take the  $q$  limit directly and obtain

$$D\chi = 2\alpha(1-u^2)^3 - i4\alpha^2(1+u^2)u^4\phi_{VV}(0, 0) \quad (2.28)$$

where  $\chi = (1+u)/(1-u)$  is the static susceptibility [=  $\tilde{C}(0)$ ]. Further-

more, since  $\lim_{q \rightarrow 0} \langle V_q R(z) \sigma_{-q} \rangle = 0$  we can replace  $\phi_{VV}(0, 0)$  by

$$C_V \equiv \langle V_0 R(0) V_0 \rangle \tag{2.29}$$

and obtain

$$D = D^{(s)}(1 - Q) \tag{2.30}$$

where

$$D^{(s)} = 2\alpha(1 + u)^2(1 - u)^4 \tag{2.31}$$

and

$$Q = i2\alpha u^4 \frac{(1 + u^2)}{(1 - u^2)^3} C_V \tag{2.32}$$

We also note that we can evaluate the memory function  $\phi(q, z)$  for all  $q$  and  $z$  in the noninteracting ( $u = 0$ ) limit as

$$\lim_{u \rightarrow 0} \phi(q, z) = 8\alpha \sin^2(q/2) \tag{2.33}$$

Notice that even in this limit there is a  $q$  dependence since there is still a dynamic coupling between spins. In this case  $\phi(q, z)$  is independent of  $z$  and we obtain a Lorentzian frequency dependence for the dynamic structure factor.

### 3. GENERAL DEVELOPMENT OF THE RSDRG

We first divide the system into cells containing two spins and associate an Ising block spin  $\mu (= \pm 1)$  with each cell. We denote the spins in cell  $i$  by  $(i, a)$ , where  $a = \pm 1$  (see Fig. 1). The renormalization group transformation mapping the original equilibrium probability distribution  $P[\sigma]$  onto the probability distribution  $P[\mu]$  governing the block spins is

$$P[\mu] = \sum_{\sigma} P[\sigma] T[\mu | \sigma] \tag{3.1}$$

where  $T[\mu | \sigma]$  is the mapping function that picks out the slowly varying

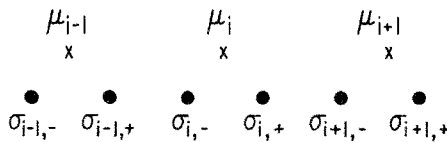


Fig. 1.  $\sigma$  Spins (solid circles) are grouped into two spins per cell.  $\mu$  Spins (crosses) are block spins associated with each cell. Convention for labeling cells and cell constituents is shown.

degrees of freedom. We will determine the mapping function  $T[\mu|\sigma]$  and the transformed SEO operator  $D[\mu|\mu']$  as usual in the RSDRG method via the eigenvalue condition<sup>(5,8)</sup>

$$\tilde{D}_\sigma T[\mu|\sigma] = D_\mu T[\mu|\sigma] \quad (3.2)$$

i.e.,  $T[\mu|\sigma]$  is an eigenvector of  $\tilde{D}[\sigma|\sigma']$  with the eigenvalue  $D[\mu|\mu']$ . We also demand that  $T[\mu|\sigma]$  satisfy the normalization conditions<sup>(8)</sup>:

$$\sum_\mu T[\mu|\sigma] = 1 \quad (3.3)$$

and

$$\sum_\sigma P[\sigma] T[\mu|\sigma] T[\mu'|\sigma] = \delta_{\mu,\mu'} P[\mu] \quad (3.4)$$

Next we define the projection of any spin variable  $A(\sigma)$  onto the block spins via

$$A(\mu)P[\mu] = \sum_\sigma P[\sigma] T[\mu|\sigma] A(\sigma) \quad (3.5)$$

Using this definition we can then define a projection operator<sup>(8)</sup>

$$\mathbb{P}A(\sigma) \equiv \sum_\mu T[\mu|\sigma] A(\mu) \quad (3.6)$$

and its complement

$$\mathbb{Q} = 1 - \mathbb{P} \quad (3.7)$$

Using this projection operator it was shown in Ref. 13 that we can rewrite a general time-dependent correlation function

$$C_{AB}(t) = \sum_\sigma P[\sigma] B(\sigma) e^{\tilde{D}_\sigma t} A(\sigma) \quad (3.8)$$

in the form

$$C_{AB}(t) = C'_{AB}(t) + \bar{C}_{AB}(t) \quad (3.9)$$

where

$$C'_{AB}(t) = \sum_\mu P[\mu] B(\mu) e^{\tilde{D}_\mu t} A(\mu) \quad (3.10)$$

is the slowly varying part of  $C_{AB}(t)$  which projects onto the block-spin degrees of freedom and

$$\bar{C}_{AB}(t) = \sum_\sigma P[\sigma] (\mathbb{Q}B(\sigma)) e^{\tilde{D}_\sigma t} (\mathbb{Q}A(\sigma)) \quad (3.11)$$

is the rapidly varying part of  $C_{AB}(t)$  which we should be able to treat using straightforward perturbation theory.



A key question at this point is whether  $\tilde{D}_\mu$  also leads to the conservation of the total spin? We can prove the following theorem quite generally. Suppose  $M(\sigma)$  is a conserved quantity

$$\tilde{D}_\sigma M(\sigma) = 0 \tag{3.12}$$

Then if  $M(\mu)$  is the projection of  $M(\sigma)$  along the block spins, as defined in (3.5), then, for a  $D_\mu$  satisfying (3.2)

$$\tilde{D}_\mu M(\mu) = 0 \tag{3.13}$$

and  $M(\mu)$  is conserved on the  $\mu$  lattice. The proof is straightforward. Multiply (3.12) by  $P[\sigma]T[\mu|\sigma]$  and sum over  $\sigma$ :

$$\sum_\sigma P[\sigma]T[\mu|\sigma]\tilde{D}_\sigma M(\sigma) = 0 \tag{3.14}$$

However, using the detailed balance<sup>(8)</sup> this becomes

$$\sum_\sigma P[\sigma]M(\sigma)\tilde{D}_\sigma T[\mu|\sigma] = 0 \tag{3.15}$$

After using (3.2) and the definition of  $M(\mu)$  we obtain

$$D_\mu P[\mu]M(\mu) = P[\mu]\tilde{D}_\mu M(\mu) = 0 \tag{3.16}$$

Consequently our coarse-graining procedure preserves the conservation law to all orders in our development.

## 4. PERTURBATION EXPANSION

### 4.1. Statics

Since we are unable to carry out the coarse-graining procedure described in the previous section exactly, we must use perturbation expansion methods for constructing the mapping function  $T[\mu|\sigma]$  and the coarse-grained operator  $D[\mu|\mu']$ . In carrying out this perturbation theory analysis we will treat the coupling between cells as a small parameter. In the cell notation the Hamiltonian can be written as

$$H[\sigma] = K_C V_C[\sigma] + K_I V_I[\sigma] \tag{4.1}$$

where

$$V_C[\sigma] = (1/2) \sum_{i,a} \sigma_{i,a} \sigma_{i,-a} \tag{4.2}$$

$$V_I[\sigma] = (1/2) \sum_{i,a} \sigma_{i,a} \sigma_{i+a,-a} \tag{4.3}$$

and we allow for different cell ( $K_C$ ) and intercell ( $K_I$ ) couplings. If we take

$K_I = K_C$  and expand in  $K_C$  we get the usual cumulant expansion. It was shown in Ref. 14 that this method leads to poor results for the static correlation functions for this problem. Hence, we will allow  $K_C$  and  $K_I$  to vary from their bulk values and assume they have an expansion in an order parameter  $\epsilon$ :

$$\begin{aligned} K_C &\rightarrow K_C(\epsilon) + \epsilon [K_c - K_c(\epsilon)] \\ K_I &\rightarrow K_I(\epsilon) + \epsilon [K_I - K_I(\epsilon)] \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} K_C(\epsilon) &\equiv K_0 + \sum_{n=1}^{\infty} \epsilon^n K_n^0 \\ K_I(\epsilon) &\equiv K_R^0 + \sum_{n=1}^{\infty} \epsilon^n K_n^R \end{aligned} \quad (4.5)$$

The parameters  $K_0$ ,  $K_n^0$ ,  $K_R^0$ ,  $K_n^R$  are to be determined through the imposition of constraints on an order by order basis. With the expansions given by (4.4) and (4.5) and treating  $V_I[\sigma]$  to be explicitly of  $O(\epsilon)$ , the Hamiltonian becomes

$$\begin{aligned} H[\sigma] &= \{K_C(\epsilon) + \epsilon [K_c - K_c(\epsilon)]\} V_C[\sigma] \\ &\quad + \epsilon \{K_I(\epsilon) + [K_I - K_I(\epsilon)]\} V_I[\sigma] \end{aligned} \quad (4.6)$$

Expanding the equilibrium probability distribution to first order in  $\epsilon$  we obtain

$$P[\sigma] = P_0[\sigma] \{1 + \epsilon \{\Delta K_1 [V_C(\sigma) - Nu_0/2] + K_R^0 V_I[\sigma]\} + O(\epsilon^2)\} \quad (4.7)$$

where

$$P_0[\sigma] = e^{H_0[\sigma]} / \sum_{\sigma'} e^{H_0[\sigma']} \quad (4.8)$$

is the zeroth-order probability distribution,

$$H_0[\sigma] = K_0 V_C[\sigma] \quad (4.9)$$

and

$$\Delta K_1 = K - K_0 + K_1^0 \quad (4.10)$$

$$u_0 = \tanh K_0 \quad (4.11)$$

In order to simplify matters and since it will influence our results very little we will choose  $K_1^0$  such that  $\Delta K_1 = 0$ . We have yet to determine  $K_0$  and  $K_R^0$ . This will be carried out in Section 6.

### 4.2. Dynamics

Here we develop the same type of perturbation expansion for the dynamic operator  $\tilde{D}_\sigma$ . In the cell notation we have

$$\tilde{D}[\sigma | \sigma'] = \alpha_C \tilde{D}_C[\sigma | \sigma'] + \alpha_I \tilde{D}_I[\sigma | \sigma'] \tag{4.12}$$

where

$$\begin{aligned} \tilde{D}_C[\sigma | \sigma'] &= -(1/2) \sum_i \Lambda_{\sigma, \sigma'}^{[i, +; i, -1]} \mathcal{W}^{[i, +; i, -1]}(\sigma) \\ &\times [\alpha_C(\sigma_{i,+} - \sigma_{i,-})(\sigma'_{i,+} - \sigma'_{i,-}) + \beta_C \sigma_{i,+} \sigma_{i,-} \sigma'_{i,+} \sigma'_{i,-}] \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \tilde{D}_I[\sigma | \sigma'] &= -(1/2) \sum_i \Lambda_{\sigma, \sigma'}^{[i, +; i+1, -1]} \mathcal{W}^{[i, +; i+1, -1]}(\sigma) \\ &\times [\alpha_I(\sigma_{i,+} - \sigma_{i+1,-})(\sigma'_{i,+} - \sigma'_{i+1,-}) + \beta_I \sigma_{i,+} \sigma_{i+1,-} \sigma'_{i,+} \sigma'_{i+1,-}] \end{aligned} \tag{4.14}$$

Note that  $\tilde{D}_{I,\sigma}$  explicitly couples cells. Consequently we must, for consistency, treat  $\tilde{D}_{I,\sigma}$  to be of  $O(\epsilon)$ . We also expand the dynamic coefficients  $\alpha_C, \alpha_I, \beta_C$  and  $\beta_I$  as in (4.4) and (4.5). We therefore introduce the parameters  $\alpha_0, \alpha_n^0, \alpha_n^R, \alpha_n^R, \beta_0, \beta_n^0, \beta_n^R, \beta_n^R$ . We can then expand the various parts of  $\tilde{D}_\sigma$  in a power series in  $\epsilon$  and obtain the explicit terms in the expansion

$$\tilde{D}_\sigma = \tilde{D}_\sigma^{(0)} + \epsilon \tilde{D}_\sigma^{(1)} + O(\epsilon^2) \tag{4.15}$$

The zeroth-order operator is

$$\begin{aligned} \tilde{D}_\sigma^{(0)} &= -\frac{1}{2} \sum_i \Lambda_{\sigma, \sigma'}^{[i, +; i, -1]} (1 - u_0 \sigma_{i,+} \sigma_{i,-}) \\ &\times [\alpha_0(\sigma_{i,+} - \sigma_{i,-})(\sigma'_{i,+} - \sigma'_{i,-}) + \beta_0 \sigma_{i,+} \sigma_{i,-} \sigma'_{i,+} \sigma'_{i,-}] \end{aligned} \tag{4.16}$$

### 5. RENORMALIZATION TO $O(\epsilon)$

Using the perturbation expansion described in Section 4 we can construct  $T[\mu | \sigma]$  and  $D_\mu$  in an expansion in  $\epsilon$ :

$$T[\mu | \sigma] = T_0[\mu | \sigma] + \epsilon T^{(1)}[\mu | \sigma] + \dots \tag{5.1}$$

$$D_\mu = D_\mu^{(0)} + \epsilon D_\mu^{(1)} + \dots \tag{5.2}$$

The eigenvalue condition (3.2) then becomes to zeroth order

$$\tilde{D}_\sigma^{(0)} T_0[\mu | \sigma] = D_\mu^{(0)} T_0[\mu | \sigma] \tag{5.3}$$

**Table I. Eigenfunctions and Eigenvalues of the Cell Operator  $\tilde{D}^{(0)}$ . Also the Projection of  $\sigma_{i,\alpha}$  Along the  $n$ th Eigenfunction**

$i$	$\psi_i(\sigma)$	$\lambda_i$	$\nu_i(a)$
0	1	0	0
1	$\frac{1}{[2(1+u_0)]^{1/2}}(\sigma_{i,+} + \sigma_{i,-})$	0	$\left(\frac{1+u_0}{2}\right)^{1/2}$
2	$\frac{1}{[2(1-u_0)]^{1/2}}(\sigma_{i,+} - \sigma_{i,-})$	$4\alpha_0(1+u_0)$	$a\left(\frac{1-u_0}{2}\right)^{1/2}$
3	$\frac{1}{(1-u_0^2)^{1/2}}(\sigma_{i,+}\sigma_{i,-} - u_0)$	$2\beta_0$	0

and to first order

$$\tilde{D}_\sigma^{(1)}T_0[\mu|\sigma] + \tilde{D}_\sigma^{(0)}T^{(1)}[\mu|\sigma] = D_\mu^{(1)}T_0[\mu|\sigma] + D_\mu^{(0)}T^{(1)}[\mu|\sigma] \quad (5.4)$$

The explicit solution of the zeroth-order problem involves the identification of the slowest varying degrees of freedom in a cell. The eigenfunctions  $\Psi_i^{(n)}$  and eigenvalues  $\lambda_n$  of the zeroth-order operator  $\tilde{D}_\sigma^{(0)}$  (4.14) satisfy

$$\tilde{D}_\sigma^{(0)}\Psi_i^{(n)}(\sigma) = -\lambda_n\Psi_i^{(n)} \quad (5.5)$$

and are normalized such that

$$\sum_\sigma P_0[\sigma]\Psi_i^{(n)}(\sigma)\Psi_j^{(m)}(\sigma) = \delta_{m,n} \quad (5.6)$$

We list the four eigenfunctions and eigenvalues in Table I. We note that  $\lambda_0 = 0$  reflects the time independence of the "vacuum" state and that  $\lambda_1 = 0$  reflects the conservation of the spin property.

It is clear that  $T_0[\mu|\sigma]$  satisfying (5.3) and mapping the slowest modes onto the block spin variables is

$$T_0[\mu|\sigma] = \prod_{i=1}^{N/2} T_0^i[\mu|\sigma] \quad (5.7)$$

where

$$T_0^i[\mu|\sigma] = (1/2)[1 + \mu_i\psi_i^{(1)}(\sigma)] \quad (5.8)$$

Then, since

$$\tilde{D}_\sigma^0 T_0[\mu|\sigma] = 0 \quad (5.9)$$

we easily obtain that

$$D^{(0)}[\mu|\mu'] = 0 \quad (5.10)$$

$T_0[\mu|\sigma]$  satisfies the normalization conditions (3.3) and (3.4). Using these and the eigenvalue condition (3.2) one easily finds from (5.4) the first-order

contribution to the renormalized operator given by

$$D^{(1)}[\mu|\mu']P_0[\mu'] = \langle T_0[\mu|\sigma]\tilde{D}_\sigma^{(1)}T_0[\mu'|\sigma] \rangle \tag{5.11}$$

Having obtained  $T_0[\mu|\sigma]$  we can now carry out the static renormalization (3.1) to  $O(\epsilon)$ . To this order the static renormalization is exactly the same as in the spin-flip case treated by Mazenko and Luscombe<sup>(14)</sup> (ML). The renormalized probability distribution turns out to be<sup>(14)</sup>

$$P[\mu] = P_0 \left[ 1 + \frac{1}{4} K_R^0 (1 + u_0) \sum_{i,a} \mu_i \mu_{i+a} \right] \tag{5.12}$$

where

$$P_0 = \frac{1}{2^{N'}} \tag{5.13}$$

$N'$  being the number of cells in the system. Comparing (5.11) with our original probability distribution we see that the first-order renormalized static parameter  $K'$  is given by

$$K' = \frac{1}{2} K_R^0 (1 + u_0) \tag{5.14}$$

We now proceed to the dynamic renormalization. We use the first-order term  $\tilde{D}_\sigma^{(1)}$ , obtained through the expansions explained in Section 4, in (5.11) and obtain the first-order renormalized operator  $\tilde{D}_\mu^{(1)}$ . It is in the same form as our original operator  $\tilde{D}_\sigma$  (2.8) with  $W^{[i,i+1]}(\mu) = 1$  and with the first-order renormalized dynamic parameters given by

$$\alpha' = \frac{(1 - u_0)^2 (1 + u_0)}{4} \alpha_R^0 \tag{5.15}$$

$$\beta' = \frac{(1 - u_0)^2}{8} (2\alpha_R^0 + \beta_R^0) \tag{5.16}$$

From (5.15) and (5.16) we get

$$R_{\beta'} = \frac{1}{2(1 + u_0)} (2 + R_\beta) \tag{5.17}$$

where  $R_{\beta'} \equiv \beta'/\alpha'$  and we assume  $R_{\beta_R^0} = R_\beta$  to first order. One can easily verify (using the  $u_0$  given in Section 6) that starting at any nonzero temperature ( $0 \leq u < 1$ ) and with a finite  $R_\beta(u)$ , the successively iterated values of  $R_\beta$  approach the fixed point:

$$R^*(u = 0) = 2 \tag{5.18}$$

If  $u = 1$ , then the iterated values approach the other fixed point:

$$R^*(u = 1) = \frac{2}{3} \tag{5.19}$$

This shows that the operator  $\tilde{D}_\beta[\sigma|\sigma']$  does not disappear under the effect of renormalization transformation.

### 6.1. Recursion Relations for Observables

The recursion relations for static observables resulting from an evaluation of coarse-grained variable using (3.5) to lowest order in  $\epsilon$  are identical to those found by ML. We recall the simple recursion relation satisfied by the magnetic susceptibility

$$\chi = 2\nu_1^2 \chi' \quad (6.1)$$

where

$$\nu_1 = \langle \sigma_{i,a} \Psi_i^{(1)}(\sigma) \rangle_0 \quad (6.2)$$

and is given in Table I.

We now turn to the dynamic quantities. We implement (3.5) for the case  $A(\sigma) = \sigma_{i,a}$ . Using (3.5) to lowest order in  $\epsilon$  we have

$$A(\mu) = \nu_1 \mu_i \quad (6.3)$$

Therefore, for the projected variable and its complement [see (3.6) and (3.7)] we have

$$\mathbb{P}\sigma_{i,a} = \sum_{\mu} T_0[\mu | \sigma] \mu_i = \nu_1 \Psi_i^{(1)}(\sigma) \quad (6.4)$$

and

$$\mathbb{Q}\sigma_{i,a} = \sum_n (1 - \delta_{1,n}) \nu_n(a) \Psi_i^{(n)}(\sigma) \quad (6.5)$$

We then use Eqs. (3.9) through (3.11) to obtain the recursion relation for  $C_{i,a;j,a}(t) \equiv \langle \sigma_{i,a} \sigma_{j,a}(t) \rangle$  to lowest order in  $\epsilon$ :

$$C_{i,a;j,a}(t) = \nu_1^2 C'_{i,j}(t) + \nu_2^2 \delta_{i,j} a a' e^{-\lambda_2 t} \quad (6.6)$$

Taking the Laplace–Fourier transform we obtain

$$C(q, z) = 2\nu_1^2 \cos^2(q/2) C'(q', z) = \frac{2\nu_2^2 \sin^2(q/2)}{z + i4\alpha_0(1 + u_0)} \quad (6.7)$$

where

$$q' = 2q \quad (6.8)$$

Another dynamic quantity of interest is  $C_V$  introduced in Section 2. Indicating the spin dependence of  $V_0$  explicitly and using the same formalism which led to equations (3.9)–(3.11) we obtain

$$C_V \equiv \langle V(\sigma) R_{\sigma} V(\sigma) \rangle = \langle V(\mu) R_{\mu} V(\mu) \rangle' + \langle \mathbb{Q}V(\sigma) R_{\sigma} \mathbb{Q}V(\sigma) \rangle \quad (6.9)$$

We evaluate the coarse-grained quantity  $V(\mu)$  using equation (3.5) to lowest order and find  $V^{(0)}(\mu) = 0$ . That means, to this order in  $\epsilon$  we have

$$C_V = \langle \mathbb{Q}V(\sigma)R_\sigma \mathbb{Q}V(\sigma) \rangle_0 = \langle V(\sigma)R_\sigma V(\sigma) \rangle_0 \equiv C_V^{(0)} \quad (6.10)$$

### 6.2. Determination of Unknown Parameters

There are a number of parameters which enter into our analysis which have not yet been determined. We have the static parameters  $K_0$ ,  $K_R^0$  and the dynamic parameters  $\alpha_0$ ,  $\alpha_R^0$ ,  $\beta_0$ , and  $\beta_R^0$ .

We determine the static parameters following the same procedure as ML, that is, we choose  $u_R^0$  such that the thermal recursion relation is given by

$$u' = u^2 \quad (6.11)$$

For a discussion of this relation see ML. We have from (5.26)

$$K_R^0 = \frac{2}{1 + u_0} K'(u) \quad (6.12)$$

where  $K'(u)$  is determined by (6.11). Then, given  $u' = u^2$ ,  $u_0$  is chosen by demanding that the recursion relation<sup>(14)</sup> for the quantity  $\epsilon = \langle \sigma_i \sigma_{i+1} \rangle$  gives the exact value  $\epsilon = u$ . The resulting  $u_0$  is

$$u_0 = \frac{u(4 - u)}{2 + u^2} \quad (6.13)$$

(6.12) and (6.13) together with (6.11) determine  $K_0(u)$  and  $K_R^0(u)$ .

Let us turn next to the determination of the dynamic parameters. In this regard we also follow the analysis of ML. The main difference here is that we must be careful in treating the conservation law. If we insert the form for  $C(q, z)$  given by (2.14) into (6.7) and take the small  $q$  and  $z$  limits we obtain

$$\frac{\chi}{2\nu^2\chi'} = \lim_{q \rightarrow 0} \lim_{z \rightarrow 0} \frac{\phi(q, z)}{\phi'(q', z)} \quad (6.14)$$

If we then use (6.1) relating  $\chi$  and  $\chi'$  at this order we obtain the simple result

$$1 = \lim_{q \rightarrow 0} \lim_{z \rightarrow 0} \frac{\phi(q, z)}{\phi'(q', z)} \quad (6.15)$$

Since

$$\phi(q, 0) = \alpha f(q, u) \quad (6.16)$$

where  $f$  is some function of  $q$  and  $u$  alone, we have

$$1 = \lim_{q \rightarrow 0} \frac{\alpha}{\alpha'} \frac{f(q, u)}{f(q', u')} \quad (6.17)$$

If we define

$$\Delta = \alpha' / \alpha \quad (6.18)$$

then

$$\Delta = \lim_{q \rightarrow 0} \frac{f(q, u)}{f(q', u')} \quad (6.19)$$

Using (2.27) we can write for small  $q$  that

$$\alpha f(q, u) = q^2 \chi^{-1} \Gamma \quad (6.20)$$

where

$$\Gamma \equiv D\chi \quad (6.21)$$

Then using (6.1) and (6.19) we obtain

$$\Delta = \frac{1}{8\nu_1^2} \frac{(\Gamma/\alpha)}{(\Gamma'/\alpha')} \quad (6.22)$$

We recall from Section 2 that

$$\Gamma = \Gamma^{(s)}(1 - Q) \quad (6.23)$$

where

$$\Gamma^{(s)} = 2\alpha(1 - u^2)^3 \quad (6.24)$$

and

$$Q = i2\alpha u^4 \frac{1 + u^2}{(1 - u^2)^3} C_V \quad (6.25)$$

We see that one needs to determine  $Q$  in order to obtain  $\Delta$ . We note first that  $C_V$  in  $Q$  is multiplied by  $u^4$ . That means if we carry out a high-temperature expansion in  $u$ , we have the first three terms, without needing to evaluate  $C_V$ . The low-temperature limit is considerably more complicated than the high-temperature limit. In particular we have not been able to extract the leading term for  $C_V$  in a low-temperature expansion. We can however determine  $C_V$  by our lowest-order renormalization analysis as explained in Section 6.1. There we found that to this order

$$C_V = \langle V(\sigma) R_\sigma V(\sigma) \rangle_0 \quad (6.26)$$



Table II.  $Q(u)$  for Various  $u$  Values

$u$	$Q$
0	0
0.1	$2.4 \times 10^{-5}$
0.2	$3.5 \times 10^{-4}$
0.3	$1.5 \times 10^{-3}$
0.4	$3.6 \times 10^{-3}$
0.5	$6.1 \times 10^{-3}$
0.6	$7.6 \times 10^{-3}$
0.7	$6.7 \times 10^{-3}$
0.8	$3.8 \times 10^{-3}$
0.9	$8.2 \times 10^{-4}$
1.0	$1.3 \times 10^{-4}$

The average in (6.26) can be easily evaluated to give

$$C_V = -\frac{i}{2} \nu_2^2 \left( \frac{4\nu_1^4}{\lambda_3} + \frac{2u_0^2\nu_2^2}{\lambda_2} + \frac{4\nu_1^2\nu_2^2}{\lambda_2 + \lambda_3} + \frac{\nu_1^4}{\lambda_2} + \frac{\nu_1^2\nu_2^2}{\lambda_2} + \frac{\nu_2^4}{3\lambda_2} \right) \quad (6.27)$$

where

$$\lambda_2 = 4\alpha_0(1 + u_0) \quad (6.28)$$

and

$$\lambda_3 = 2\beta_0 \quad (6.29)$$

As will be explained later in this section,  $\alpha_0$  and  $\beta_0$  depend on  $\Delta$  and therefore on  $Q$ , through Eqs. (6.22) and (6.23). This means that the  $C_V$  in Eq. (6.25) depends on  $Q$  itself. This is not a severe limitation since, as we shall see in a self-consistent manner,  $Q$  is small for all temperatures. As a first step we choose  $Q = 0$ . This gives  $\Delta = (\Gamma^{(s)}/\alpha)/(\Gamma'^{(s)}/\alpha') \equiv \Delta^{(s)}$ . Then we evaluate  $C_V$  through Eq. (6.26) using  $\Delta^{(s)}$  in the expressions that give  $\alpha_0$  and  $\beta_0$ <sup>4</sup> as a function of temperature. The resulting function  $Q$  is shown in Table II for various temperatures. We note that  $Q$  is never larger than 0.008 over the entire temperature scale. Therefore it is an excellent approximation to take  $Q = 0$  in the equation of  $\Delta$ . Thus we will use

$$\Delta = \frac{1}{8\nu_1^2} \frac{\Gamma^{(s)}/\alpha}{\Gamma'^{(s)}/\alpha'} = \frac{1}{8\nu_1^2(1 + u^2)^3} \quad (6.30)$$

in the rest of our calculations.

<sup>4</sup>The dependences of  $\alpha_0$  and  $\beta_0$  on  $\Delta$  can be seen through equations (6.31)–(6.32) and (6.31)–(6.34), respectively.

Having determined  $\Delta(u)$  we then have from (5.15) that

$$\alpha_R^0 = \frac{4\Delta\alpha}{(1-u_0)^2(1+u_0)} \quad (6.31)$$

Consider now the cell dynamic parameter  $\alpha_0$ . Since  $\alpha_R^0$  is an intercell parameter it should be smaller than  $\alpha_0$  for the consistency of our perturbation expansion. However,  $\alpha_R^0$  diverges as  $u$  goes to 1. For simplicity we assume that

$$\alpha_0 = C\alpha_R^0 \quad (6.32)$$

where  $C$  is a temperature-independent constant. We determine  $C$  so that the same site dynamic correlation function (see Section 7) gives [Eq. (7.18)] the correct value for the diffusion coefficient at  $u = 0$ . This yields  $C = 2.56$ .

We can now determine  $\beta_0$  and  $\beta_R^0$ . It is reasonable to assume that to lowest order in  $\epsilon$  we have

$$\frac{\beta_R^0}{\alpha_R^0} = \frac{\beta_0}{\alpha_0} = \frac{\beta}{\alpha} \equiv R_\beta \quad (6.33)$$

We have shown in Section 5 that  $R_\beta = 2$  at the high-temperature fixed point  $u = 0$ . Since  $u$  always flows to this point in our iterative solutions of the recursion relations we will take

$$R_\beta = 2 \quad (6.34)$$

Then, using Eqs. (6.31)–(6.34) we obtain the temperature dependences of the coefficients  $\beta_0$  and  $\beta_R^0$ .

## 7. ANALYSIS OF THE RECURSION RELATIONS

Since the static recursion relations have been studied extensively in ML, we focus here strictly on dynamical results. Writing the recursion relation (6.7) in terms of the Fourier transformed correlation function  $C(q, \omega)$  we have

$$C(q, \omega) = \frac{2\nu_1^2}{\Delta} \cos^2(q/2)C'(q', \omega') + \frac{32\nu_1^2\nu_2^2\alpha_0 \sin^2(q/2)}{\omega^2 + 64\alpha_0^2\nu_1^4} \quad (7.1)$$

where the renormalized frequency  $\omega'$  is given by

$$\omega' = \frac{\omega}{\Delta} \quad (7.2)$$

Let us first look at the diffusion coefficient  $D$ . It is related to  $C(q, \omega)$  by

$$D\chi = \lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \frac{\omega^2}{q^2} C(q, \omega) \quad (7.3)$$

Using (7.1) and (7.3) we find that the diffusion coefficient satisfies the

recursion relation

$$D = 4D' \tag{7.4}$$

Writing

$$D = \alpha F[u]$$

and using  $\Delta = \alpha'/\alpha$  we have

$$F(u) = 4\Delta F(u') \tag{7.5}$$

Clearly this homogeneous recursion relation has a solution  $F(u)$  which is undetermined up to a multiplicative constant. Since as  $u \rightarrow 0$ ,  $\Delta \rightarrow 1/4$  we have

$$F(0) = F'(0) \tag{7.6}$$

and we can fix this constant by using the exact high-temperature limit

$$F(0) = 2 \tag{7.7}$$

We note, since  $\Delta$  is given by the ratio

$$\Delta = \frac{\chi^{-1}(\Gamma^{(s)}/\alpha)}{4(\chi')^{-1}(\Gamma^{(s)}/\alpha')} \tag{7.8}$$

that a solution of the recursion relation (7.5) is

$$D = \chi^{-1}\Gamma^{(s)} = D^{(s)} \tag{7.9}$$

Direct numerical iteration of the recursion relation leads to the same result.

Another important dynamical quantity of interest is the dynamical exponent  $z$ . Dynamical scaling states that  $t' = b^{-z}t$ , where  $b = 2$  is the rescaling factor. We also have  $t'/t = \alpha'/\alpha = \Delta$  and  $\Delta$  has already been determined in the previous section. Thus we use (6.30) in

$$z = \lim_{u \rightarrow 1} (\ln \Delta / \ln 2) \tag{7.10}$$

and find

$$z = 6 \tag{7.11}$$

We note that (7.11) is consistent with the condition

$$z \geq 5 \tag{7.12}$$

found by Haake and Thol.<sup>(23)</sup> In the model introduced by Zwerger the  $V_q$  term (2.20) in  $\tilde{D}_\sigma \sigma_q$  (2.19) does not appear. As a consequence, the dynamic memory function  $\phi^{(d)}$ , while not zero, is proportional to  $q^4$  for small  $q$ . The static memory function  $\phi^{(s)}$  is known exactly and is proportional to  $q^2$ . Thus the small  $q$  limit of the memory function, in particular the diffusion coefficient (2.27), can be determined exactly to be

$$D = 2\alpha(1 - u)^3(1 + u^2) \tag{7.13}$$

On comparing this with our diffusion coefficient (2.31), we note that (7.13) goes as  $\xi^{-3}$  for low temperatures while (2.31) has the behavior  $\xi^{-4}$ . [ $\xi \sim (1-u)^{-1}$  is the correlation length.] For this reason, the dynamical exponent  $z = 5$  obtained from (7.13) is different from our value  $z = 6$ .

Let us next consider the same site time correlation function. Setting  $i = j$  and  $a = a'$  in (6.6) we obtain

$$C_0(t) = \nu_1^2 C'_0(t) + \nu_2^2 e^{-\lambda_2 t} \quad (7.14)$$

We iterate (7.14) numerically. The results are shown in Fig. 2 for  $0 < \alpha t < 5.0$ . The effect of the critical slowing down (as  $u \rightarrow 1$ ) is clearly seen. We can write

$$C_0(t) = \int_{-\pi}^{+\pi} \frac{dq}{2\pi} C(q, t) \quad (7.15)$$

For long times ( $\alpha t \gg 1$ ) we expect that only the wave numbers that are in the hydrodynamical region ( $q\xi \ll 1$ ,  $\xi$  is the correlation length) will contribute to the integral and we can replace  $C(q, t)$  with its hydrodynamical form:

$$C(q, t) = \chi e^{-Dq^2 t} \quad (7.16)$$

so that

$$C_0(t) = \frac{\chi}{(4\pi Dt)^{1/2}} \quad (7.17)$$

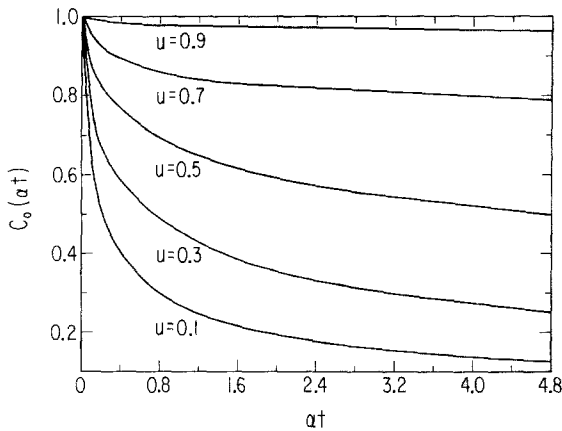


Fig. 2. Time dependence of the spin-spin autocorrelation function from the RSDRG at various temperatures.

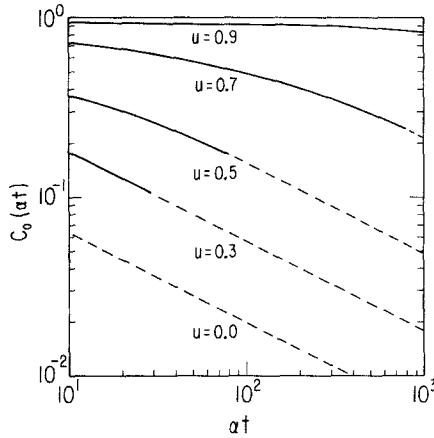


Fig. 3. Time dependence of  $C_0(\alpha t)$  for long times. The dashed lines represent the approximate hydrodynamic regions.

for large  $t$ . For shorter times, the wave-number values that are in the critical region ( $q\xi \gg 1$ ) will also be important and (7.17) will no longer hold. We note that the onset of the hydrodynamical regime is temperature dependent. Therefore the transition of  $C_0(t)$  to the hydrodynamical region, that is, to the region of  $n = -1/2$  power law decay, will start at different time values for different temperatures. This effect is seen clearly in Fig. 3. When we fit the numerical solution of (7.1) to the form

$$C_0(t) = \frac{\chi}{2[\pi D(u)t]^{n(u)}} \tag{7.18}$$

for intermediate to large times, we obtain the correct exponent  $n = 1/2$  and the diffusion coefficient  $D(u) = D^{(s)}(u)$ . We now consider the frequency- and wavenumber-dependent correlation function recursion relation (7.1). We iterate (7.1) again using the exactly known  $u = 0$  limit as the boundary condition. In Fig. 4 we show  $C(q, \omega)$  against  $\omega$  at various temperatures and for small values of  $q$ . Also shown in the same figure is  $C^{(s)}(q, \omega)$  obtained by using the exactly known static memory function and neglecting the dynamic memory function:

$$C^{(s)}(q, \omega) = \frac{16(1 - u^2)^3 \sin^2(q/2)}{\omega^2 + 64(1 - u^2)^6 \sin^4(q/2) \tilde{C}^{-2}(q)} \tag{7.19}$$

Inspection of Fig. 4 shows that for small values of  $q$  and  $\omega$  and especially at high temperatures  $C(q, \omega)$  has a Lorentzian shape and is close to  $C^{(s)}(q, \omega)$ .

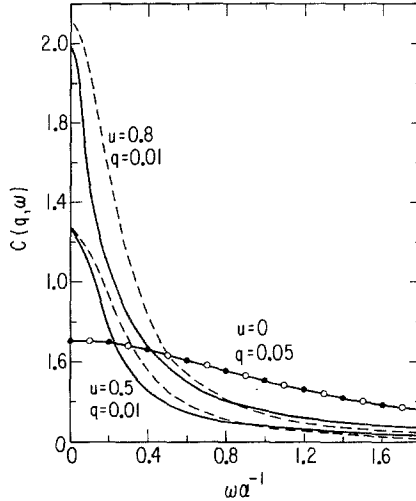


Fig. 4. Frequency dependence of the correlation function for the  $u$  and  $q$  values indicated. The solid lines and filled circles represent the RSDRG results, while the dashed lines and open circles denote  $C^{(s)}(q, \omega)$ . The values on the  $C(q, \omega)$  and  $\omega\alpha^{-1}$  axes should be multiplied by  $k_1(u)$  and  $k_2(u)$ , respectively.  $k_1(0) = 0.25 \times 10^{-1}$ ,  $k_2(0) = 0.25 \times 10^2$ ,  $k_1(0.5) = 10^{-5}$ ,  $k_2(0.5) = 10^3$ ,  $k_1(0.8) = 0.25 \times 10^{-5}$ ,  $k_2(0.8) = 0.25 \times 10^5$ .

This can also be seen in Fig. 5 where we plot the peak values  $C(q, 0)$  and  $C^{(s)}(q, 0)$  against  $q$ . We note that for  $q$  values  $q < 0.01$ , that is, in the hydrodynamic regime, the behavior of  $C(q, \omega)$  is basically identical to that of  $C^{(s)}(q, \omega)$ . The major difference between  $C(q, \omega)$  and  $C^{(s)}(q, \omega)$  arises only in the region  $0.1\pi < q < 1.0\pi$ ,  $0 < \omega\alpha^{-1} < 0.2$ . As seen in Fig. 5 the peak value of  $C(q, \omega)$  gradually becomes greater than that of  $C^{(s)}(q, \omega)$  and  $C(q, \omega)$  starts developing an additional peak located around  $\omega = 0$ , as we go from small to high values of  $q$ . This peak persists to  $u = 0$  (near  $q = \pi$ ). We know, however, from the exact solution at  $u = 0$  that it is an artifact of our approximation. The source of this spurious peak can be easily seen in an analysis of (7.1). The point is that the term proportional to  $\sin^2(q/2)$  in (7.1) generates a very strong frequency dependence near  $\omega = 0$  owing to the rescaling  $\omega \rightarrow \omega/\Delta$  under normalization. If  $\omega$  is not small, and since  $1/\Delta > 4$ , one finds after a few iterations that one scales to very large frequencies and the  $\sin^2(q/2)$  term scales to small values. One does not obtain this effect if  $\omega$  is very small. Consequently we generate this peak near  $\omega = 0$ . This effect appears to be tied to a piece of physics neglected in our treatment of this problem. The peak is associated with a very small frequency rescaling factor  $\Delta$ . This rescaling factor is physically sensible in treating small wave numbers. It seems reasonable, however, that this

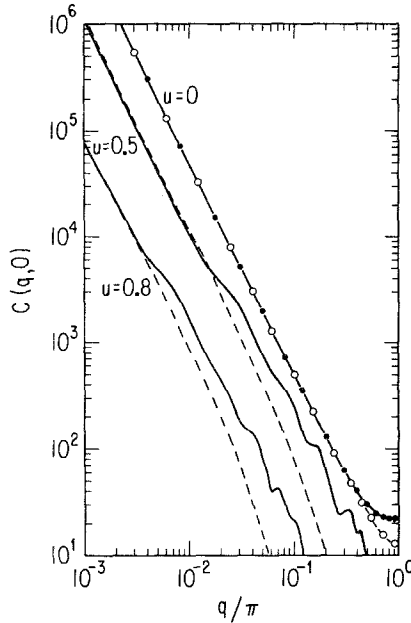


Fig. 5. Wave-number dependence of  $C(q, 0)$  for the  $u$  values indicated. The filled circles are the RSDRG results, while the open circles are  $C^{(s)}(q, 0)$ . The  $C(q, 0)$  axis should be multiplied by  $k(u)$ .  $k(0) = 10^{-2}$ ,  $k(0.5) = 1.0$ ,  $k(0.8) = 10^3$ .

rescaling factor, which reflects a conserved mode at  $q = 0$ , is inappropriate for describing the time rescaling for wave numbers near  $q = \pi$ . In order to remedy this inadequacy in the theory one should go back and carry through the analysis where one keeps track of the variables associated with the antiferromagnetic points and determine the appropriate time rescaling for these variables.

In conclusion we point out that our treatment here is rather unsophisticated. We have worked at the lowest order in perturbation theory and treated cells with only two spins. Consequently our results are rather rough, particularly for intermediate values of wave numbers. We believe, however, that we know how to systematically improve these results. Our primary purpose in this paper has been to show that we can develop a sensible RSDRG analysis for treating dynamic systems with a conserved variable.

## ACKNOWLEDGMENTS

One of us would like to acknowledge support from an Alfred P. Sloan Foundation Fellowship and from NSF Grant No. DMR 77-12637. E. O.

acknowledges the support from a N.A.T.O. Science Fellowship provided by the Scientific and Technical Research Council of Turkey.

## REFERENCES

1. Y. Achiam and J. Kosterlitz, *Phys. Rev. Lett.* **41**:128 (1978).
2. G. F. Mazenko, M. Nolan, and O. T. Valls, *Phys. Rev. Lett.* **41**:500 (1978).
3. Y. Achiam, *J. Phys.* **A11**:L129 (1978).
4. W. Kinzel, *Z. Phys. B* **29**:361 (1978).
5. G. F. Mazenko, in *Dynamical Critical Phenomena and Related Topics*, C. P. Enz, ed. (Springer-Verlag, Berlin, 1979).
6. M. Suzuki, K. Sogo, I. Matsuba, H. Ikeda, T. Chikama, and H. Takano, *Prog. Theor. Phys.* **61**:864 (1979).
7. S. T. Chui, G. Forgacs, and H. L. Frisch, *Phys. Rev. B* **20**:243 (1979).
8. G. F. Mazenko, M. J. Nolan, and O. T. Valls, *Phys. Rev. B* **22**:1275 (1980).
9. G. F. Mazenko, M. J. Nolan, and O. T. Valls, *Phys. Rev. B* **22**:1263 (1980).
10. U. Decker and F. Haake, *Z. Phys. B* **36**:379 (1980).
11. S. Ma, *Phys. Rev. B* **19**:4824 (1979).
12. G. F. Mazenko, J. Hirsch, M. J. Nolan, and O. T. Valls, *Phys. Rev. Lett.* **44**:1083 (1980).
13. G. F. Mazenko, J. Hirsch, M. J. Nolan, and O. T. Valls, *Phys. Rev. B* **23**:1431 (1981).
14. G. F. Mazenko and J. Luscombe, *Ann. Phys. (N. Y.)*, **132**:121 (1980).
15. G. F. Mazenko and O. T. Valls, *Phys. Rev. B* **24**:1404 (1981).
16. G. F. Mazenko and O. T. Valls, in *Real Space Renormalization*, J. M. J. van Leeuwen and T. Burkhardt, eds. (Springer-Verlag, Berlin, to be published).
17. K. Kawasaki, *Phys. Rev.* **145**:224 (1966).
18. K. Kawasaki, in *Phase Transitions and Critical Phenomena*, Vol. 2, C. Domb and M. S. Green, eds. (Academic Press, New York, 1972).
19. W. Zwerger, *Phys. Lett.* **84A**(5):269 (1981).
20. K. Binder, *Monte Carlo Methods in Statistical Physics* (Springer-Verlag, New York, 1979).
21. K. Binder, *Z. Phys.* **267**:313 (1974).
22. R. Glauber, *J. Math. Phys.* **4**:294 (1963).
23. F. Haake and K. Thol, *Z. Phys. B* **40**:219 (1980).